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A Note on Business Cycle Accounting*

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Abstract

Chari, Kehoe, and McGrattan (2007) (CKM) show that a large class of dynamic stochastic general equilibrium (DSGE) models with various frictions and shocks is observationally equivalent to a benchmark real business cycle (RBC) model with correlated "wedges" in the RBC model's first-order conditions. The wedges in the static first-order conditions of the RBC model can be readily computed by evaluating the first-order conditions at the data and then solving for the wedges. In contrast, identification of the "investment wedge" in the RBC model's dynamic Euler equation requires the researcher to make assumptions about the expectation formation by agents in the RBC model. In particular, CKM assume that expectations are formed as if, from the perspective of the model's agents, wedges followed a vector autoregressive process of order one (VAR(1)). We show that wedges generally do not have a VAR(1) representation, implying that CKM's procedure is based on model-inconsistent expectations. We also provide an alternative, model-consistent approach to modeling expectation formation. On the former issue, we present a necessary and sufficient "rank condition" under which a detailed economy can be mapped into a benchmark model where wedges follow a VAR(1) process. On the latter issue, we suggest that the information set underlying the expectation formation should not only contain current wedges, but also all predetermined variables.

JEL-Classification: C50, E10

Keywords: Business Cycle Accounting, Model Consistent Expectations

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1 Introduction

Chari, Kehoe, and McGrattan (2007) (CKM) show that a large class of dynamic stochastic general equilibrium (DSGE) models with various frictions and shocks is observationally equivalent to a benchmark real business cycle (RBC) model with correlated "wedges" in the RBC model's first-order conditions. Since different DSGE models - CKM refer to these models as "detailed economies" - have different implications for the dynamic properties of the wedges, the wedges reveal information about the structure of the unkown data generating economy.

The wedges in the static first-order conditions of the RBC model can be readily computed by evaluating the first-order conditions at the data and then solving for the wedges. In contrast, identification of the "investment wedge" in the RBC model's dynamic Euler equation requires the researcher to make assumptions about the expectation formation by agents in the RBC model. In particular, CKM assume that expectations are formed as if, from the perspective of the model's agents, wedges followed a vector autoregressive process of order one (VAR(1)). We argue that this assumption is inappropriate for some interesting and widely discussed detailed economies.[1]

We argue further that the VAR(1) assumption is not only critical for computing the investment wedge but also for implementing the accounting as proposed by CKM. Indeed, the impact on equilibrium quantities might be wrongly assessed even for correctly measured wedges.

This can be seen as follows. In order to assess the contribution of different wedges to business cycle movements, CKM suggest to set the values of the other wedges to constants, leaving the distribution of the wedges of interest unchanged. They then calculate the decision rules as functions of the operating wedges, their expected future values and the predetermined variables in the RBC model. Finally, they plug the measured wedges and their expected future values, as obtained from the VAR(1), into the decision rules in order to get simulated equilibrium quantities which they compare to the data.

We show that wedges generally do not have a VAR(1) representation, implying that CKM's procedure is based on model-inconsistent expectations. This result holds independently of whether the wedges were correctly measured in the first place. We also provide an alternative, model-consistent approach to modeling expectation formation. On the former issue, we present a necessary and sufficient "rank condition" under which a detailed economy can be mapped into a benchmark model where

[1] One example is mentioned in the critique of the accounting procedure by Christiano and Davies (2006).

wedges follow a VAR(1) process. On the latter issue, we suggest that the information set underlying the expectation formation should not only contain current wedges, but also all predetermined variables.

We illustrate our results for the sticky wage model discussed in CKM. For that model, the rank condition is not satisfied, implying that the accounting procedure proposed by CKM is inconsistent with the assumption of rational expectations. We also show that a simple application of our proposal - augmenting the VAR(1) of the wedges by capital - resolves these problems.

2 Rank Condition

Suppose that the solution of the detailed rational expectations model can be written in the following state-space form

$$\begin{aligned} c_t &= M_{cp} p_t + M_{ce} e_t && (1) \\ \underbrace{\begin{pmatrix} p_t \\ e_t \end{pmatrix}}_{:=S_t} &= \underbrace{\begin{pmatrix} N_{pp} & N_{pe} \\ 0 & \rho \end{pmatrix}}_{:=\Theta} \begin{pmatrix} p_{t-1} \\ e_{t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \sigma \end{pmatrix}}_{:=\Sigma} \varepsilon_t && (2) \end{aligned}$$

where c_t is a vector of non-predetermined, p_t a vector of predetermined variables[2] and e_t an exogenous vector autoregressive process of order 1 (with serially uncorrelated innovations ε_t).

Let W_t be the vector of the wedges needed for mapping this model into the benchmark RBC model of CKM. In order to understand how the wedges are related to the state variables S_t of the detailed economy, one has to plug the solved equilibrium processes (in closed form) of the detailed economy into the linearized first order conditions of the benchmark RBC model. The wedges which distort the static first order conditions can then directly be written as a linear combination of the states p_t and e_t. This is not so clear for the investment wedge which distorts the Euler equation because this equation involves both, the current and the expected future investment wedge. However, this equation can be solved forward and also the investment wedge turns up to be a linear combination of p_t and e_t. Hence, the closed form solution of the wedges is given by

$$W_t = Z_p p_t + Z_e e_t.$$

Note that when solving the expectational equation for the investment wedge forward,

[2]In the sense of Blanchard and Kahn (1980), i.e. $\mathbb{E}_{t-1} p_t = p_t$.

we implicitely use the correct specification of the expectation. This is no longer the case when the expactations are infered from a VAR(1) in the wedges when these do not have a VAR(1) representation.

We now discuss when the wedges have a VAR(1) representation. Using equation (2), the process of the wedges can be written as

$$W_t = Z \begin{pmatrix} p_t \\ e_t \end{pmatrix} = ZS_t = Z(\Theta S_{t-1} + \Sigma\varepsilon_t) \tag{3}$$

where

$$Z = \left(\begin{array}{ccc} Z_p & \vdots & Z_e \end{array} \right)$$

Theorem 1. *Assuming that the detailed economy, described by (1) and (2), maps into a benchmark RBC model with wedges $W_t = ZS_t$, then the process of the wedges has a VAR(1) representation, i.e. $\mathbb{E}_{t-1}[W_t - \Phi W_{t-1}] = 0$ where $\mathbb{E}_{t-1}$ is the expectation conditional on all information up to time $t-1$, if and only if*

$$rank \begin{pmatrix} Z' & \vdots & \Theta' Z' \end{pmatrix} = rank(Z) \tag{4}$$

Proof. We show that the condition is necessary (step i) and sufficient (step ii).

i) Plugging in (3) into $\mathbb{E}_{t-1}[W_t - \Phi W_{t-1}] = 0$ yields

$$\mathbb{E}_{t-1}[Z\Theta S_{t-1} - \Phi Z S_{t-1}] = 0.$$

Almost surely, it follows that

$$Z\Theta = \Phi Z. \tag{5}$$

Equation (5) states that each row of $Z\Theta$ lies in the row space of Z. Since the dimension of the row space of Z is equal to $rank(Z)$, (4) follows.

ii) Given that the rank condition (4) is verified, it follows that the product $\Theta' Z'$ lies in the column space of Z'. Hence, there exists a matrix Φ such that

$$Z\Theta = \Phi Z.$$

Since, by assumption,

$$W_t = Z\Theta S_{t-1} + Z\Sigma\varepsilon_t,$$

it follows that

$$W_t = \Phi W_{t-1} + Z\Sigma\varepsilon_t.$$

$\square$

Theorem 2. *Assuming that the detailed economy, described by (1) and (2), maps into a benchmark RBC model with wedges $W_t = ZS_t$ and that the vector of predetermined variables in the benchmark model, k_t, is the same as in the detailed economy, then $W_t^k := \begin{pmatrix} k_t' & W_t' \end{pmatrix}'$ has a VAR(1) representation if Z_e is invertible.*

Proof. By assumption

$$\begin{pmatrix} k_t \\ W_t \end{pmatrix} = \begin{pmatrix} I & 0 \\ Z_p & Z_e \end{pmatrix} \begin{pmatrix} p_t \\ e_t \end{pmatrix}$$

Since it is assumed that the inverse of Z_e exists, it follows

$$\begin{pmatrix} I & 0 \\ -Z_e^{-1} Z_p & Z_e^{-1} \end{pmatrix} \begin{pmatrix} k_t \\ W_t \end{pmatrix} = \begin{pmatrix} p_t \\ e_t \end{pmatrix}$$

and the VAR(1) representation follows from equation (2). $\square$

3 Example: Sticky Wages

CKM present a sticky wage economy which is observationally equivalent to a benchmark model with a labor wedge given by

$$W_t = 1 - \tau_{L,t} = -\frac{U_{L_t}}{U_{C_t}} \frac{1}{F_{L_t}}$$

where U_{L_t} (resp. U_{C_t}) is the marginal utility of labor (resp. consumption) and F_{L_t} is the marginal productivity of labor. The labor wedge captures the distortions between the marginal product of labor and the marginal rate of substitution between leisure and consumption. The detailed economy is driven by a stochastic money supply, which is called e_t, and the only predetermined variable is capital k_t. Since the solution to the detailed economy is a linear combination of capital and the money supply shock, the first order accurate dynamics of the labor wedge is determined by the underneath state space system.

$$\begin{aligned} 1 - \tau_{L,t} &= \begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} k_t \\ e_t \end{pmatrix} \\ \begin{pmatrix} k_t \\ e_t \end{pmatrix} &= \begin{pmatrix} n_{kk} & n_{ke} \\ 0 & \rho \end{pmatrix} \begin{pmatrix} k_{t-1} \\ e_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t \end{aligned}$$

where $Z = \begin{pmatrix} z_1 & z_2 \end{pmatrix}$. Since $rank(Z) = 1$, the rank condition of Theorem 1 is not satisfied for meaningful calibrations:

$$rank\begin{pmatrix} Z' & \vdots & \Theta' Z' \end{pmatrix} = rank\left(\begin{pmatrix} z_1 & n_{kk} z_1 \\ z_2 & n_{ke} z_1 + \rho z_2 \end{pmatrix}\right) \leq 2.$$

The misspecification by imposing a VAR(1) in the wedges may lead to wrong accounting results. This is the case even if the realized wedges are correctly measured. The reason is that the solution of the benchmark model depends on the process of the wedges. If the process is wrong, also the (rational expectation) solution to the model is wrong.[3]

Following Theorem 2, there is a simple solution to this potential misspecification: Writing

$$\begin{pmatrix} k_t \\ 1 - \tau_{L,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1 & z_2 \end{pmatrix} \begin{pmatrix} k_t \\ e_t \end{pmatrix}$$

it can be seen that the assumptions of Theorem 2 are satisfied if $z_2 \neq 0$, which is the case for most calibrations of the parameters in the detailed economy. Hence, by not restricting the correlations between the labor wedge and capital to be zero in the estimation, we mitigate the need for estimating an infinite order VAR or VARMA process for the wedges.

4 Conclusion

We have derived a necessary and sufficient condition for the existence of a VAR(1) representation of the wedges. We have then shown that for the sticky wage model of CKM, this condition is not satisfied. Hence, we conclude that the model is not representable in the form that CKM estimate.

We suggest an extended econometric model that allows to accurately estimate the dynamics of the wedges. The solution is based on the fact that there is a VAR(1) representation in the vector of the wedges augmented by the capital stock.

Obviousely, this extension does not provide a solution for all DSGE models proposed in the literature. For example, if the stochastic money supply is replaced by an interest rate rule with interest rate smoothing, then the lagged interest rate is an

[3] In the language of CKM, p. 797, the decision rules are not correctly computed. Hence, the realized sequences of output, labor and investment and therefore also of the capital stock are not correctly computed.

additional predetermined variable which does not have a counterpart in the benchmark RBC. In this case, the vector with the wedges and capital does not have a VAR(1) representation. However, one could generalize the benchmark RBC model by adding predetermined variables such that there is a VAR(1) representation in the wedge vector augmented with the predetermined variables for a larger class of detailed economies.

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